Theoretical Evaluation of Lossless / Lossy Wavelet Transforms under Lossless / Lossy Unified Coding Gain

Masahiro Iwahashi* Pongsak Mahachoklertwattana** Jarree Chaicharn** Noriyoshi Kambayashi*
*Department of Electrical Engineering, Nagaoka University of Technology, Nagaoka, Niigata, 940-2188, JAPAN
**Department of Electrical Engineering, Thammasat University, Khlong-Luang, Pathum-Thani 12121, THAILAND

Abstract

This paper proposes a new measure referred to the *"lossless / lossy unified coding gain"* for the purpose of theoretical evaluation of some *"reversible wavelets"*. The reversible wavelet is recently proposed as a new coding method, which can be used not only as a lossless coding but also as a lossy coding. It is also convenient because of its progressive functionality. So far some reversible wavelets have been proposed as JPEG-2000 candidates. This paper evaluates them in lossless mode and in lossy mode respectively under the new objective measure.

1. INTRODUCTION

Digital image data compression technique has been playing an important role in the area of information communications. JPEG international standard is one of the most widely used algorithm in the world to compress a huge amount of digital image data.

The conventional JPEG [1] adopts two different modes - lossless mode and lossy mode. Toward the year of 2000, a new movement to establish a new international standard (JPEG-2000) has been set up aiming at a lossless / lossy unified coding method [2]. As it works not only as a lossless but also as a lossy encoder, we call it *"lossless / lossy wavelets"* (LLW) here and after.

So far, some LLW are proposed as the JPEG-2000 candidates [3]. Some researchers have been investigating the LLW in respect of lossless coding performance [3,4]. However their conclusions are based on some practical experiments without theoretical evaluation.

In this paper, we define a new objective measure -*"lossless / lossy unified coding gain"*- and theoretically evaluate the LLW in lossless mode and in lossy mode respectively. The measure introduced here can be used to optimize parameters of a coding system [7].

2. LOSSLESS / LOSSY WAVELET

In this section, we summarize the LLW to be theoretically compared in **4.** with the measure defined in **3.**.

2.1 Encoding

Analysis part of the LLW maps integer input image data into integer de-correlated data. Synthesis part can reproduce the same image data as the original without any loss. Introducing quantization into the LLW, it also works as a lossy encoder.

Figure 1 illustrates a "double lifting" structured wavelet transform [5]. Defining z-transform of the original signal $x(k) \in$ integer, $k=0,1,\cdots,K-1$, as

$$X(z)=\sum_{k=0}^{K-1} x(k) z^{-k} \quad (1)$$

X(z) is divided into two groups by

$$\begin{pmatrix} X_0(z) \\ X_1(z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ z^{1/2} & -z^{1/2} \end{pmatrix}\begin{pmatrix} X(z^{1/2}) \\ X(-z^{1/2}) \end{pmatrix} \quad (2)$$

After this, FIR filters $P_{01}(z)$ and $P_{10}(z)$ are applied to produce two band signals $Y_0(z)$ and $Y_1(z)$ by

$$\begin{pmatrix} Y_0(z) \\ Y_1(z) \end{pmatrix} = \begin{pmatrix} X_0(z) \\ X_1(z) \end{pmatrix} + \begin{pmatrix} R[P_{10}(z)\cdot Y_1(z)] \\ R[P_{01}(z)\cdot X_0(z)] \end{pmatrix} \quad (3)$$

where

$$R[X(z)]=\sum_{k=0}^{K-1} \lfloor x(k)+0.5 \rfloor z^{-k}$$

These band signals are entropy coded and embedded into a bit-stream in lossless mode. On the other hand, in lossy mode, the band signals are quantized with step size ($\alpha_b S_Y$) before the entropy coding, where S_Y is a constant and two parameters α_b, b=0,1, are determined in **3.2**.

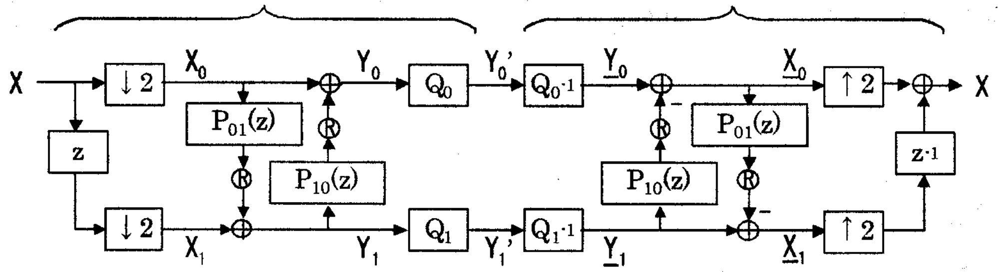

Figure 1 Lossless / lossy wavelet transform (LLW).

$$\begin{pmatrix} Y'_0(z) \\ Y'_1(z) \end{pmatrix} = \begin{pmatrix} R[Y_0(z)/(\alpha_0 S_Y)] \\ R[Y_1(z)/(\alpha_1 S_Y)] \end{pmatrix} \quad (4)$$

2.2 Decoding

The input signal is decoded by the inverse procedure of equations (2)-(4) given by

$$\begin{pmatrix} \underline{Y}_0(z) \\ \underline{Y}_1(z) \end{pmatrix} = \begin{pmatrix} Y'_0(z)\cdot(\alpha_0 S_Y) \\ Y'_1(z)\cdot(\alpha_1 S_Y) \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} \underline{X}_0(z) \\ \underline{X}_1(z) \end{pmatrix} = \begin{pmatrix} \underline{Y}_0(z) \\ \underline{Y}_1(z) \end{pmatrix} - \begin{pmatrix} R[P_{10}(z)\cdot \underline{Y}_1(z)] \\ R[P_{01}(z)\cdot \underline{X}_0(z)] \end{pmatrix} \quad (6)$$

$$\underline{X}(z) = \underline{X}_0(z^2) + z^{-1}\underline{X}_1(z^2) \quad (7)$$

From equations (2)-(7), it is clear that the reconstructed signal $\underline{X}(z)$ is exactly same as the original $X(z)$ in case of no quantization in which $\underline{Y}_b(z)=Y_b(z)$ for $b=0,1$.

2.3 Equivalent expression

Figure 1 is equivalent to figure 2 when the effect of R[ ] in equation (3) is negligible. The FIR filters $H_b(z)$, $G_b(z)$, $b=0,1$, in figure 2 are related to $P_{01}(z)$, $P_{10}(z)$ in figure 1 by

(i) 1 stage (B=2)

$$\begin{pmatrix} H_0(z) & G_0(z) & w_0 \\ H_1(z) & G_1(z) & w_1 \end{pmatrix} = \begin{pmatrix} S_0(z) & T_0(z) & 2 \\ S_1(z) & T_1(z) & 2 \end{pmatrix} \quad (8)$$

where

$$\begin{pmatrix} S_0(z) \\ S_1(z) \end{pmatrix} = \begin{pmatrix} 1 & P_{10}(z^2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P_{01}(z^2) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} T_0(z) \\ T_1(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P_{10}(z^2) & 1 \end{pmatrix} \begin{pmatrix} 1 & -P_{01}(z^2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} \quad (10)$$

The FIR filters $P_{01}(z)$ and $P_{10}(z)$ are given in **2.5**.

2.4 Octave decomposition

It is known that further decomposition of the low band signal is effective for some images. In this case, the LLW is expressed by the general expression in figure 2 with the following equations.

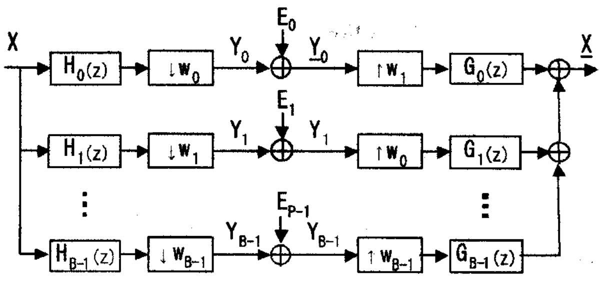

Figure 2 Equivalent expression of the LLW.

Table 1 Some of JPEG-2000 candidates [3].

name	a_0 a_1 a_2	b_0 b_1 b_2
(2, 2)	[-1]/ 2	[1]/ 4
(4, 2)	[-9 1]/ 16	[1]/ 4
(2, 4)	[-1]/ 2	[19 -3]/ 64
(4, 4)	[-9 1]/ 16	[9 -1]/ 32
(6, 2)	[-150 25 -3]/256	[1]/ 4

(ii) 2 stages (B=3)

$$\begin{pmatrix} H_0(z) & G_0(z) & w_0 \\ H_1(z) & G_1(z) & w_1 \\ H_2(z) & G_2(z) & w_2 \end{pmatrix} = \begin{pmatrix} S_0(z)\begin{pmatrix} S_0(z^2) \\ S_1(z^2) \end{pmatrix} & T_0(z)\begin{pmatrix} T_0(z^2) \\ T_1(z^2) \end{pmatrix} & \begin{matrix} 4 \\ 4 \end{matrix} \\ S_1(z) & T_1(z) & 2 \end{pmatrix} \quad (11)$$

(iii) 3 stages (B=4)

$$\begin{pmatrix} H_0(z) & G_0(z) & w_0 \\ H_1(z) & G_1(z) & w_1 \\ H_2(z) & G_2(z) & w_2 \\ H_3(z) & G_3(z) & w_3 \end{pmatrix} = \quad (12)$$

$$\begin{pmatrix} S_0(z)\begin{pmatrix} S_0(z^2)\begin{pmatrix} S_0(z^4) \\ S_1(z^4) \end{pmatrix} \\ S_1(z^2) \end{pmatrix} & T_0(z)\begin{pmatrix} T_0(z^2)\begin{pmatrix} T_0(z^4) \\ T_1(z^4) \end{pmatrix} \\ T_1(z^2) \end{pmatrix} & \begin{matrix} 8 \\ 8 \\ 4 \end{matrix} \\ S_1(z) & T_1(z) & 2 \end{pmatrix}$$

2.5 JPEG-2000 candidates [3]

Expressing the filters $P_{01}(z)$ and $P_{10}(z)$ in figure 1 as

$$\begin{pmatrix} P_{01}(z^2) \\ P_{10}(z^2) \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ b_0 & b_1 & b_2 & \cdots \end{pmatrix} \begin{pmatrix} (z+z^{-1}) \\ (z^3+z^{-3}) \\ (z^5+z^{-5}) \\ \vdots \end{pmatrix} \quad (13)$$

filter coefficients of some JPEG-2000 candidates to be evaluated in **4.** are summarized in table 1.

3. UNIFIED CODING GAIN

In this section, we define the *"lossless / lossy unified coding gain"* as a new objective measure. The LLW in table 1 are theoretically compared under this measure in **4.**.

3.1 Lossless coding gain

In lossless mode, the total bit rate B_{LSL} [bpp] of the band signals $Y_b(z)$, $b=0,1,\cdots,B-1$, are given by

$$B_{LSL} = \sum_{b=0}^{B-1} w_b^{-1} B_{Y_b}, \quad \sum_{b=0}^{B-1} w_b^{-1} = 1 \quad (14)$$

where B_{Y_b} denotes bit rate of $Y_b(z)$. Since a bit rate is estimated by the signal's variance [6], namely,

$$B_{Y_b} = \log_2 \gamma_{Y_b} \sqrt{\sigma^2_{Y_b}}, \quad b = 0,1,\cdots,B-1 \tag{15}$$

$$B_{PCM} = \log_2 \gamma_X \sqrt{\sigma^2_X} \tag{16}$$

equation (14) becomes

$$B_{LSL} = \log_2 \prod_{b=0}^{B-1} \left(\gamma_{Y_b} \sqrt{\sigma^2_{Y_b}}\right)^{W_b^{-1}} \tag{17}$$

In this report, we define a new criterion to evaluate system's performance by

$$G_{LSL} = 20 \log_{10} \frac{2^{B_{PCM}}}{2^{B_{LSL}}} = G^*_{LSL} + c_1 \tag{18}$$

where

$$G^*_{LSL} = 10 \log_{10} \frac{\sigma^2_X}{\prod_{b=0}^{B-1} (\sigma^2_{Y_b})^{W_b^{-1}}} \tag{19}$$

$$c_1 = 20 \log_{10} \frac{\gamma_X}{\prod_{b=0}^{B-1} (\gamma_{Y_b})^{W_b^{-1}}} \tag{20}$$

Equations (19),(20) are driven by (14)-(18). We refer to G^*_{LSL} in equation (19) as the *"lossless coding gain"*.

3.2 Optimum bit allocation [2]

In lossy mode, the quantization step size ($\alpha_b S_Y$), b=0,1, ⋯, B-1, should be determined to minimize the coding distortion defined by

$$\sigma^2_{LSY} = \frac{1}{K} \sum_{k=0}^{K-1} \{\underline{x}(k) - x(k)\}^2 \tag{21}$$

under a given total bit rate B_{LSY}

$$B_{LSY} = \sum_{b=0}^{B-1} w_b^{-1} B'_{Y_b} \tag{22}$$

where B'_{Y_b} is a bit rate of quantized band signals $Y_b(z)$. Substituting equations (14),(17),(22),(23),

$$B'_{Y_b} = B_{Y_b} - \log_2 (\alpha_b S_Y) \tag{23}$$

$$\sigma^2_{LSY} = \sum_{b=0}^{B-1} w_b^{-1} \|G_b\|^2 \cdot \frac{(\alpha_b S_Y)^2}{12} \tag{24}$$

where

$$\|G_b\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |G_b(e^{j\omega})|^2 d\omega \tag{25}$$

into (24), we can find that the optimization problem above is equivalent to minimize the variance in equation (26).

$$\sigma^2_{LSY} = \frac{2^{-2B_{LSY}}}{12} \cdot \theta \cdot \prod_{b=0}^{B-1} \left(\gamma_{Y_b} \sqrt{\sigma^2_{Y_b}}\right)^{2w_b^{-1}} \tag{26}$$

where

$$\Omega = 10 \log_{10} \theta = 10 \log_{10} \frac{\sum_{b=0}^{B-1} (\alpha_b^2) w_b^{-1} \cdot \|G_b\|^2}{\prod_{c=0}^{B-1} (\alpha_c^2)^{W_c^{-1}}} \tag{27}$$

A solution [9] to this problem is given by

$$\alpha_b = \sqrt{\frac{\|G_0\|^2}{\|G_b\|^2}}, \quad b = 0,1,\cdots,B-1 \tag{28}$$

3.3 Lossless / lossy unified coding gain

We show relation of the *"lossless"* coding gain in equation (19) and the *"lossy"* coding gain [6] defined by

$$G_{LSY} = 10 \log_{10} \frac{\sigma^2_{PCM}}{\sigma^2_{LSY}} \tag{29}$$

σ^2_{PCM} is a variance of quantization error with step size S_X where

$$\sigma^2_{PCM} = \frac{S_X^2}{12} \tag{30}$$

$$B_{LSY} = B_{PCM} - \log_2 S_X \tag{31}$$

Therefore

$$\sigma^2_{PCM} = \frac{1}{12} \left\{ \frac{2^{B_{PCM}}}{2^{B_{LSY}}} \right\}^2 \tag{32}$$

Substituting equations (19), (26), (32) into (29), we get

$$G_{LSY} = G^*_{LSY} + c_1 \tag{33}$$

where

$$G^*_{LSY} = G^*_{LSL} - \Omega \tag{34}$$

We define G^*_{LSY} in equation (34) as the *"lossless / lossy unified coding gain"* and we use this measure to evaluate LLW in lossy mode.

4. SIMULATION

In this section, the new objective measure defined in **3.** is verified and performance of the JPEG-2000 candidates are theoretically evaluated.

4.1 AR(1) model and criterion

In this report, we use AR(1) model with correlation coefficient ρ whose frequency characteristics is

$$|X(e^{j\omega})| = \frac{1-\rho}{\sqrt{1+\rho^2 - 2\rho \cdot \cos\omega}} \tag{35}$$

Here, we summarize the measures defined in this report as

$$G^*_{LSL} = 10 \log_{10} \prod_{b=0}^{B-1} \left(\|F_b\|^2\right)^{W_b^{-1}} \tag{36}$$

$$\Omega_{org} = 10 \log_{10} \sum_{b=0}^{B-1} \|G_b\|^2 w_b^{-1} \tag{37}$$

$$\Omega_{opt} = 10 \log_{10} \prod_{b=0}^{B-1} \left(\|G_b\|^2\right)^{W_b^{-1}} \tag{38}$$

where

$$\|F_b\|^2 = \frac{\sigma_X^2}{\sigma_{Y_b}^2} = \frac{\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega}{\int_0^{2\pi} |H_b(e^{j\omega}) X(e^{j\omega})|^2 d\omega} \tag{39}$$

Equation (36) is the same as (19). Equations (37) and (38) are derived from (27), substituting $\alpha_b=1$ for $b=0,1,\cdots,B-1$, and (28) respectively. Since equation (18) implies

$$\Delta B_{LSL} = -\frac{\log_2 10}{20}\cdot\Delta G_{LSL} \approx -0.166\cdot\Delta G_{LSL} \quad (40)$$

improvement in respect of bit rate is evaluated by

$$\begin{pmatrix}\Delta g_{LSL}\\ \Delta g_{LSY}\end{pmatrix} = -\frac{\log_2 10}{20}\begin{pmatrix}\Delta G_{LSL}\\ \Delta G_{LSY}\end{pmatrix} \approx -0.1661\begin{pmatrix}\Delta G_{LSL}\\ \Delta G_{LSY}\end{pmatrix} \quad (41)$$

4.2 Lossless coding gain

Table 2 indicates a result of theoretical evaluation of lossless mode with the measure defined in **3.1**. The figures in the parenthesis indicate bit-rate-improvement in equation (41) from "1 stage" for each method. To verify rightness of the measure, we attached a practical result in table 3. The total bit rate B_{LSL} for a practical AR(1) sequence is indicated. We used the entropy rate defined by

$$B_{ENT} = -\sum_{s=0} P_s \log_2 P_s \quad [bit/pixel] \quad (42)$$

where P_s denotes probability of a symbol "s" [6]. We can confirm good agreement between these two results in respect of the bit-rate-improvement.

4.3 Lossy coding gain

Unified coding gain G^*_{LSY} in equation (34) for AR(1) model is illustrated in figure 3. $G^*_{LSY(org)}$ and $G^*_{LSY(opt)}$ are the gains with Ω_{org} in equation (37) and with Ω_{opt} in equation (38) respectively. It indicates that significance of the optimization in **3.2** is +1.7 [dB] and +0.4 [dB] in average for "3 stage" and "1 stage" respectively. These are -0.28 [bpp] and -0.07 [bpp] according to equation (41).

Since the coding system works as a lossless coder and also as a lossy one, it should be evaluated in both of the two modes. Comparing G^*_{LSL} and $G^*_{LSY(opt)}$ in figure 3 (a), we can find that the method (4,2) is the best in respect of G^*_{LSL}, however, (4,4) is the best according to $G^*_{LSY(opt)}$. Similarly, (2,4) is the worst in G^*_{LSL} but (6,2) is the worst in $G^*_{LSY(opt)}$.

Table 2 A result of theoretical evaluation.
G^*_{LSL} and Δg_{LSL} for AR(1). ρ=0.9, equation (35).

	1 stage	2 stages	3 stages
(2, 2)	4.90 (0.00)	6.58 (-0.28)	7.05 (-0.36)
(4, 2)	4.87 (0.00)	6.60 (-0.29)	7.12 (-0.37)
(2, 4)	4.88 (0.00)	6.52 (-0.27)	6.97 (-0.35)
(4, 4)	4.87 (0.00)	6.59 (-0.29)	7.11 (-0.37)
(6, 2)	4.82 (0.00)	6.55 (-0.29)	7.10 (-0.38)

Table 3 A result of practical simulation.
B_{LSL} and ΔB_{LSL} for AR(1). ρ=0.9, 4096 points.

	1 stage	2 stages	3 stages
(2, 2)	6.53 (0.00)	6.24 (-0.28)	6.16 (-0.37)
(4, 2)	6.53 (0.00)	6.23 (-0.30)	6.14 (-0.38)
(2, 4)	6.52 (0.00)	6.25 (-0.27)	6.17 (-0.36)
(4, 4)	6.52 (0.00)	6.24 (-0.28)	6.15 (-0.37)
(6, 2)	6.54 (0.00)	6.25 (-0.29)	6.14 (-0.39)

5. CONCLUSION

In this report, 1) we defined the "lossless / lossy unified coding gain" as a new objective measure for LLW, and 2) theoretically evaluated JPEG-2000 candidates with this new criterion. The measure can be utilized for design of a new LLW [7]. The evaluation in this report is applicable to some other LLW such as SP transform [8] considering extrapolation and local decoding.

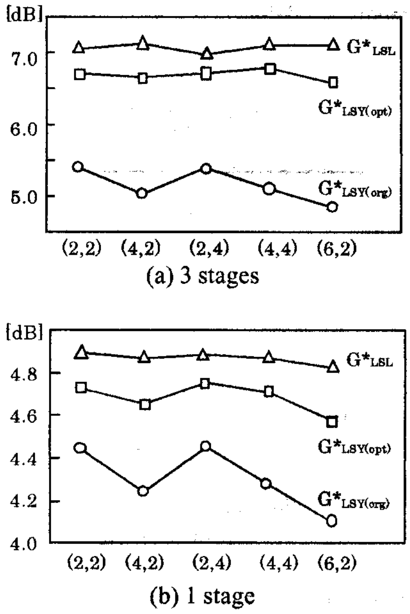

Figure 3 Unified coding gain of the LLW in table 1.

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